2.28. Simplification, Transformation, and the 3D Method

1. From DNF to CNF (and Back). Since both DNF and CNF can match every possible truth table, every truth will have a matching DNF sentence and a matching CNF sentence. So each DNF has a logically equivalent CNF sentence (and vice versa). But currently our only way of starting with one sentence and finding its equivalent mate is by detour through the appropriate truth table. Here we set out a logical equivalence encountered earlier which serves to translate from DNF to CNF (or vice versa) without appeal to truth tables.

The equivalence in question is **Distribution**.[1]

Distribution

$(\underline{\bullet \lor} (\blacktriangle \land \ast))$ **is equivalent to** $((\underline{\bullet \lor} \blacktriangle) \land (\underline{\bullet \lor} \ast))$

$(\underline{\bullet \land} (\blacktriangle \lor \ast))$ **is equivalent to** $((\underline{\bullet \land} \blacktriangle) \lor (\underline{\bullet \land} \ast))$

To transform a DNF sentence into a CNF sentence we read the upper equivalence left to right – as in this example.

$$(\underline{{\sim}P \lor} (Q \land R)) \Rightarrow ((\underline{{\sim}P \lor} Q) \land (\underline{{\sim}P \lor} R))$$

In the next case multiple applications of Distribution take us from DNF to CNF.

$$(\ \underline{(P \land Q) \lor} (R \land {\sim}S) \Rightarrow (\ \ (\underline{(P \land Q) \lor} R) \land (\ \underline{(P \land Q) \lor} {\sim}S)\ \ )$$

$$(\ \ ((P \land Q) \underline{\lor R}) \land (\ (P \land Q) \underline{\lor {\sim}S})\ \ ) \Rightarrow$$

$$(\ \ ((P \underline{\lor R}) \land (Q \underline{\lor R})) \land ((P \underline{\lor {\sim}S}) \land (Q \underline{\lor {\sim}S}))\ \ )$$

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[1] This equivalence was first stated in 2.17.1 Problem C1a – C2b; see also Problems A2, A3, A10, and B3a-B4b.

Translating from CNF to DNF reads the lower equivalence from left to right.

$$(\underline{\bullet \wedge} (\blacktriangle \vee \ast)) \textbf{ is equivalent to } ((\underline{\bullet \wedge} \blacktriangle) \vee (\underline{\bullet \wedge} \ast))$$

The following transformation illustrates.

$$(\underline{P \wedge} (\sim Q \vee R)) \Rightarrow ((\underline{P \wedge} \sim Q) \vee (\underline{\sim P \wedge} \sim R))$$

Distribution changes the size of the **scope** of the wedge and vel: starting with a sentence where the wedge has wider scope, we end with an equivalent sentence where the vel has wider scope (or vice versa).

2. **The 3D Method**. In an earlier reading we noted that we could translate from valuation sentences to anti-valuation sentences, or vice versa, through appeal to the equivalences **Double Negation** and **DeMorgan's Law**.[2]

Double Negation	**De Morgan's Law**
$\sim\sim\bullet \equiv \bullet$	$\sim(\bullet \vee \blacktriangle) \equiv (\sim\bullet \wedge \sim\blacktriangle)$
	$\sim(\bullet \wedge \blacktriangle) \equiv (\sim\bullet \vee \sim\blacktriangle)$

Using these two equivalences along with Distribution, we can transform any sentence in the Chapter Two language into an equivalent DNF sentence – which, as we've just seen, can be transformed in turn into a CNF equivalent. For obvious reasons we call this transformation procedure the **3D Method**.

[2] In 2.26 §3.

The Chapter Two language features these 13 types of sentences – with those not allowed in DNF in boldface.

1. Sentence Letters
2. Negations of Sentence Letters
3. **Negations of Negations**
4. **Negations of Conjunctions**
5. **Negations of Disjunctions**
6. Conjunctions of Sentence Letters
7. Conjunctions of Negations
8. Conjunctions of Conjunctions
9. **Conjunctions of Disjunctions**
10. Disjunctions of Sentence Letters
11. Disjunction of Negations
12. Disjunctions of Conjunctions
13. Disjunctions of Disjunctions

For each forbidden form we wield one of our three equivalences.

Non-DNF Sentence	**Transformation Rule**
3. Negation of Negation	**Double Negation (DN)**
4. Negation of Conjunction	**DeMorgan’s Law (DM)**
5. Negation of Disjunction	**DeMorgan’s Law (DM)**
9. Conjunction of Disjunction(s)	**Distribution (D)**

An example illustrates the 3D Method at work.

1. $\sim((P \vee Q) \wedge \sim\sim(R \vee (S \wedge T)))$
2. $\sim((P \vee Q) \wedge \underline{\sim\sim}(R \vee (S \wedge T))) \Rightarrow \sim((P \vee Q) \wedge (R \vee (S \wedge T)))$ DN
3. $\underline{\sim}((P \vee Q) \underline{\wedge} (R \vee (S \wedge T))) \Rightarrow (\underline{\sim}(P \vee Q) \underline{\vee} \underline{\sim}(R \vee (S \wedge T)))$ DM
4. $(\underline{\sim}(P \underline{\vee} Q) \vee \sim(R \vee (S \wedge T))) \Rightarrow ((\underline{\sim}P \underline{\wedge} \underline{\sim}Q) \vee \sim(R \vee (S \wedge T)))$ DM
5. $((\sim P \wedge \sim Q) \vee \underline{\sim}(R \underline{\vee} (S \wedge T))) \Rightarrow$
   $((\sim P \wedge \sim Q) \vee (\underline{\sim}R \underline{\wedge} \underline{\sim}(S \wedge T)))$ DM
6. $((\sim P \wedge \sim Q) \vee (\sim R \wedge \underline{\sim}(S \underline{\wedge} T))) \Rightarrow$
   $((\sim P \wedge \sim Q) \vee (\sim R \wedge (\underline{\sim}S \underline{\vee} \underline{\sim}T)))$ DM
7. $(((\sim P \wedge \sim Q) \vee (\underline{\sim R \wedge} (\sim S \vee \sim T))) \Rightarrow$
   $((\sim P \wedge \sim Q) \vee ((\underline{\sim R \wedge} \sim S) \vee (\underline{\sim R \wedge} \sim T)))$ D

And here's a second example.

1. ~((P ∧ Q) ∨ (~R ∧ S))
2. (~(P ∧ Q) ∧ ~(~R ∧ S)) 1, DM
3. ((~P ∨ ~Q) ∧ ~(~R ∧ S)) 2, DM
4. ((~P ∨ ~Q) ∧ (~~R ∨ ~S)) 3, DM
5. ((~P ∨ ~Q) ∧ (R ∨ ~S)) 4, DN
6. (((~P ∨ ~Q) ∧ R) ∨ ((~P ∨ ~Q) ∧ ~S)) 5, D
7. (((~P ∧ R) ∨ (~Q ∧ R)) ∨ ((~P ∨ ~Q) ∧ ~S)) 6, D
8. (((~P ∧ R) ∨ (~Q ∧ R)) ∨ ((~P ∧ ~S) ∨ (~Q ∧ ~S))) 7, D

Further applications of Distribution take any DNF sentence into CNF. For instance, Sentence (8) from that last example maps into Sentence (5) by three applications of Distribution. So the 3D Method transforms any Chapter Two sentence into both a DNF and a CNF equivalent – all without appeal to truth tables.

3. Simplification: Disjunctions. Beyond transforming a sentence into DNF or CNF, we can use further logical equivalences to simplify formal sentences.

For simplifying disjunctions, we begin with the following central principle.

If ● entails ▲ , then (● ∨ ▲) is equivalent to ▲ .

For example: since "(P ∧ Q)" entails "P", "((P ∧ Q) ∨ P)" is equivalent to "P".[3]

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[3] The equivalence of ((P ∧ Q) ∨ P) and "P" is one of the **absorption** laws. See 2.17.1 Problem C 4A.

Since the disjunction is equivalent to "P", we can replace the disjunction with "P" without making any semantic change. So in general:

D1. If ● entails ▲ , we can simplify disjunction (● ∨ ▲) to just ▲.

One application of this central principle provides a second maneuver for simplifying disjunctions: since **a contradiction entails any and every sentence**[4], if one part of the disjunction is a contradiction, the whole disjunction is equivalent to its other part. For example: since "$(P \wedge \sim P)$" is a contradiction, "$((P \wedge \sim P) \vee Q)$" is equivalent to "Q". That yields the following principle.

D2. If ● is a contradiction, we can simplify disjunction (● ∨ ▲) to ▲.

As a special case of that last principle, note that if both parts of a disjunction are contradictions, by (D2) one of those contradictions can be deleted – leaving just the other contradiction – without semantic change. Hence the original disjunction was equivalent to a contradiction.

D3. If both parts of a disjunction are contradictions, the whole disjunction is a contradiction.

A further application of our original principle comes via the fact that **any sentence entails itself**.[5] So if a disjunction has the same sentence as both its left and right parts, the whole disjunction is equivalent to that sentence.

D4. We can simplify disjunction (● ∨ ●) to just ●.

For example, "$(P \vee P)$" can be simplified to just "P".[6]

[4] As noted in 2.19 §1.

[5] As noted in 2.19 §3.

[6] The equivalence of "$(P \vee P)$" and "P" is one of the principles of **idempotence**. See 2.17.1 Problem C 3A.

4. Simplification: Conjunctions. All of those points have parallels involving conjunctions. Our central principle here will be the following.

If ● entails ▲ , then (●∧ ▲) is equivalent to ● .

For example: since “P” entails “(P ∨ Q),” “((P ∨ Q) ∧ P)” is equivalent to “P”.[7]

Since the conjunction is equivalent to “P”, we can replace the conjunction with “P” without no semantic change. So in general:

C1. If ● entails ▲ , we can simplify conjunction (● ∧ ▲) to just ●.

An application of this principle provides a second rule for simplifying conjunctions: since **a tautology is entailed any and every sentence**[8], if one part of the conjunction is a tautology, the whole conjunction is equivalent to its other part. For example, since “(P ∨ ~P)” is a tautology, “((P ∨ ~P) ∧ Q)” is equivalent to “Q”. That yields the following principle of simplification.

C2. If ● is a tautology, we can simplify conjunction (● ∧ ▲) to just ▲.

As an application of this last principle, note that if both parts of a conjunction are tautologies, one tautology can be deleted without semantic change to the sentence. Hence the original conjunction is logically equivalent to the remaining tautology.

C3. If both parts of a conjunction are tautologies, the whole conjunction is a tautology.

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[7] The equivalence of ((P ∧ Q) ∨ P) and “P” is the other **absorption** law. See 2.17.1 Problem C 4B.

[8] As noted in 2.19 §4.

A further application of our core principle comes (again) from the fact that **any sentence entails itself**. If a conjunction has the same sentence as both its left and right parts, the whole conjunction is equivalent to that sentence.

> **C4. We can simplify conjunction (● ∧ ●) to just ●.**

For example, “(P ∧ P)” can be simplified to just “P”.[9]

5. Application: A New Test. Finally, let us add two observations concerning valuation conjunctions and valuation disjunctions, in order to tie together the above transformation principles into a novel test of tautology, contradiction, and validity.

First, note that if one of the parts of a disjunction is a tautology, the whole disjunction is a tautology. For example, “((P ∨ ~P) ∨ Q)” is a tautology, since its left part “(P ∨ ~P)” is. That yields the following principle of transformation.

> **D5. If ● is a tautology, we can simplify disjunction (● ∨ ▲) to just ●.**

As an application of this, note that a **basic disjunction** – a however-many-place disjunction of basics – is a tautology if (and only if) it contains both a sentence letter and the negation of that sentence letter.

> **D6. A basic disjunction is a tautology if (and only if) it contains both a sentence letter and its negation**.

The following is an example.

> ((((P ∨ Q) ∨ ~R) ∨ S) ∨ ~P)

Since order and grouping of parts is semantically irrelevant in a however-many-barreled disjunction, we can re-arrange the order of parts in this sentence to group together “P” and “~P”.

[9] The equivalence of “(P ∧ P)” and “P” is the other half of **idempotence**. See 2.17.1 Problem C 3B.

$$(((\underline{(P \lor {\sim}P)} \lor {\sim}R) \lor S) \lor Q)$$

The whole disjunction is equivalent to "$(P \lor {\sim}P)$," making it a tautology. By contrast, the following basic disjunction, not containing any sentence letter and its negation, isn't a tautology.

$$((((P \lor Q) \lor {\sim}R) \lor S) \lor {\sim}P)$$

The corresponding principle for conjunctions is that if one of the parts of a conjunction is a contradiction, the whole conjunction is a contradiction. For example, "$((P \land {\sim}P) \land Q)$" is a contradiction since its left part "$(P \land {\sim}P)$" is. That yields the following principle of transformation.

> **C5. If ● is a contradiction, we can simplify conjunction (● ∧ ▲) to just ●**.

Here too we can apply the principle to **basic conjunctions** – however-many-place conjunctions of basics.

> **C6. A basic conjunction is a contradiction if (and only if) it contains a sentence letter and its negation**.

The following sentence, for example, is a contradiction, since the whole conjunction is equivalent to "$(P \land {\sim}P)$".

$$((((\underline{P} \land Q) \land {\sim}R) \land S) \land \underline{{\sim}P})$$

These points about basic disjunctions and basic conjunctions are relevant because **a DNF sentence is a disjunction of basic conjunctions**, and **a CNF sentence is a conjunction of basic disjunctions**. To see whether a formal sentence is a contradiction, we need only transform it into DNF and see if each of its parts contains both a sentence letter and its negation. Likewise, to see whether a sentence is a tautology, we need only transform the sentence into CNF and see if each of the parts of the resulting sentence contains both a sentence letter and its negation.

But thanks to the 3D Method, we can transform **any** Chapter Two sentence into DNF and CNF. So we can transform the sentence in question into both DNF and CNF. The DNF test of its constituent basic conjunctions tells us whether or not the whole sentence is a contradiction, while its CNF counterpart tells whether the sentence is a tautology.

Finally – recalling the link between validity and inconsistency[10] – we can use DNF to determine the validity of an argument **with no appeal to semantics**.

We begin with the **Counterexample Sentence** – the conjunction of all the premise and the negation of the conclusion – and transform this into DNF. Since each part of a DNF sentence is a basic conjunction, we apply the contradiction test to each of those parts (following principle **C6**). If each valuation conjunction is inconsistent (containing some sentence letter and its negation), then by (**D3**) the entire DNF sentence is inconsistent. And the Counterexample Sentence is inconsistent if (and only if) the corresponding argument is valid – as in the following two (familiar) examples.

1. $(P \lor Q)$
2. $\sim P$

$\therefore$ Q

1. $((P \lor Q) \land \sim P) \land \sim Q)$
2. $(((P \land \sim P) \lor (Q \land \sim P)) \land \sim Q)$
3. $(((\cancel{\underline{P} \land \underline{\sim P}) \land \sim Q}) \lor ((\cancel{\underline{Q} \land \sim P) \land \underline{\sim Q}}))$

VALID

1. $(P \lor Q)$
2. P

$\therefore$ Q

1. $((P \lor Q) \land P) \land \sim Q)$
2. $(((P \land P) \lor (Q \land P)) \land \sim Q)$
3. $(((P \land P) \land \sim Q) \lor (\cancel{(\underline{Q} \land P) \land \underline{\sim Q}}))$

INVALID

[10] Discussed in 2.20 §1.

Summary:

3D Method:

Double Negation (DN)

$$\sim\sim\bullet \Rightarrow \bullet$$

DeMorgan's Law (DM)

$$\sim(\bullet \wedge \blacktriangle) \Rightarrow (\sim\bullet \vee \sim\blacktriangle)$$

$$\sim(\bullet \vee \blacktriangle) \Rightarrow (\sim\bullet \wedge \sim\blacktriangle)$$

Distribution (D)

$$(\underline{\bullet \wedge} (\blacktriangle \vee \ast)) \Rightarrow ((\underline{\bullet \wedge} \blacktriangle) \vee (\underline{\bullet \wedge} \ast))$$

$$(\underline{\bullet \vee} (\blacktriangle \wedge \ast)) \Rightarrow ((\underline{\bullet \vee} \blacktriangle) \wedge (\underline{\bullet \vee} \ast))$$

Non-DNF Sentence	**Transformed by:**
Negation of Negation	**DN**
Negation of Conjunction	**DM**
Negation of Disjunction	**DM**
Conjunction of Disjunction(s)	**D**

Distribution transforms a DNF sentence into a CNF sentence (and vice versa).

Simplification: Disjunctions:

D1. If ● entails ▲ , we can simplify (● ∨ ▲) to ▲.

D2. If ● is a contradiction, we can simplify (● ∨ ▲) to ▲.

D3. If both parts of a disjunction are contradictions, the whole disjunction is a contradiction.

D4. We can simplify disjunction (● ∨ ●) to ●.

D5. If ● is a tautology, we can simplify (● ∨ ▲) to ●.

D6. A basic disjunction is a tautology if (and only if) it contains both a sentence letter and its negation.

Simplification: Conjunctions:

C1. If ● entails ▲ , we can simplify (● ∧ ▲) to ●.

C2. If ● is a tautology, we can simplify (● ∧ ▲) to ▲.

C3. If both parts of a conjunction are tautologies, the whole conjunction is a tautology.

C4. We can simplify conjunction (● ∧ ●) to ●.

C5. If ● is a contradiction, we can simplify (● ∧ ▲) to ●.

C6. A basic conjunction is a contradiction if (and only if) it contains a sentence letter and its negation.